General Quantum Measurement: An Equilibrium Gas Model

Ting Yu and Ian C. Percival

Department of Physics, Queen Mary and Westfield College
University of London, Mile End Road, London E1 4NS, UK

Abstract

We overcome one of Bell's objections to 'quantum measurement' by generalizing the definition to include systems outside the laboratory. According to this definition a quantum measurement takes place when the value of a classical variable is influenced significantly by an earlier state of a quantum system. The quantum measurement can then take place in equilibrium systems, provided the classical motion is chaotic. This is illustrated by an 'Arnold gas' model.

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1 Introduction

In one of his last articles [1], John Bell made three charges against quantum 'measurement'. The third of these was: " In the beginning natural philosophers tried to understand the world around them.. . . Experimental science was born. But experiment is a tool. The aim remains: to understand the world. To restrict quantum mechanics to be exclusively about piddling laboratory operations is to betray the great enterprise. A serious formulation [of quantum mechanics] will not exclude the big world outside the laboratory."

This paper answers this charge by extending the definition of quantum measurement into that big world, with particular emphasis on equilibrium systems [2]. Traditionally quantum measurements take place in the laboratory, but the laboratory is only part of our universe, and all such measurements start out as imitations of natural phenomena. Cloud chambers were based on the physics of clouds, which are natural detectors of charged particles. Spark chambers imitate lightning. We can generalize *quantum measurement* to mean any process whereby the state of a quantum system influences the value of a classical variable [3, 4]. This definition then applies to the big world.

We take the view of those experimenters in the laboratory who never have any doubt that their apparatus is classical or that quantum mechanics must be used for the internal dynamics of an atom. In this paper, we restrict our attention to Bell's charge against 'quantum measurement' from this viewpoint. We are not concerned with other fundamental issues of the quantum measurement problem [6, 7, 8, 9, 10, 11, 12, 13].

Laboratory quantum measurements include particle states producing the droplets in cloud chambers, bubbles in bubble chambers and sparks in spark chambers [5]. They include photon states producing silver grains in photographic emulsions, and also electrons and photons producing electron avalanches in solid state detectors and photomultipliers.

Other quantum measurements include ions producing water droplets in clouds, photon states sending impulses through the optic nerves of owls and the states of cosmic rays that produced small but very long-lived dislocations in mineral crystals in the Jurassic era. This takeover of the physics of laboratory quantum measurement into the world outside the laboratory is here generalized, and one of the questions we have to ask is how far this generalization can go. Where else do we find quantum measurements according to this definition? In particular, are there quantum measurements in equilibrium systems?

This problem cannot be solved without a better understanding of the classical theory of equilibrium systems, in particular the influence of motion at the atomic scale on variables that are normally considered to be classical, like sound waves at audio frequencies, represented by Fourier components of the density of a gas, which is represented by a model 'Arnold gas'.

The paper is organized as follows. In Sec. II, the ideas underlying our model are given. Subsequently, in Sec. III, we introduce the *Arnold gas* model, and discuss some of its properties. It is shown that this model can be solved analytically. In Sec. IV, we present a detailed analysis of the changes in the Fourier components of the particle density of the molecules in phase space as a result of collisions which are crucial for a quantitative

description of fluctuation of the gas density. We conclude in Sec. V that there is a sense in which there are quantum measurements in equilibrium gases.

2 Equilibrium gases

Laboratory systems used for quantum measurement are very complicated physical systems, even stripped down to their bare essentials. They involve amplification in one form or another, and so do the natural systems that they imitate.

A gas in equilibrium is simpler, yet we give an example to show that quantum measurement can take place there also, in the more general sense introduced above. The reason is that the motion of the molecules in the gas is chaotic, and small changes now result in large changes later. In particular changes at the quantum level now produce significant classical fluctuations in the density later. However, unlike earlier examples, we cannot use the classical density fluctuations to learn anything specific about these earlier quantum states, because the chaos causes mixing [14], which effectively obscures the signal.

In the nineteenth century, Rayleigh recognized that these classical density fluctuations would scatter light, and that the scattering was strongly dependent on the wavelength of the light. The result is the blue of the sky. The growth of droplets of water around the charged particles produced by cosmic rays in the atmosphere is a quantum measurement. According to the theory of this paper, so are the density fluctuations in the atmosphere that cause the sky to be blue where there are no clouds. So if you ever look at the sky, as every physicist sometimes should, whether it is clear or overcast, you are seeing one example or another of quantum measurement.

3 A soluble model

In order to understand the general quantum measurement outside the laboratory, it is useful to make a detailed analysis of a model. We consider a classical *Arnold Gas* which can be analytically solved. In this model, the interaction between two molecules is represented by the Arnold cat map. We are interested in the change of Fourier components of probability density at time $t = T$ due to the initial changes of the state of gas at earlier time $t = 0$. We show that a small change in the state of a single particle produces a significant fluctuation after a finite time.

3.1 Collisions and subsystems

Our model consists of a gas of molecules. We want to find the change in the state of the gas at a time $t = T$ due to an earlier change in the coordinate and momentum of a single particle P_0 at time $t = 0$. In order to get a solvable model, some simplifications and idealizations must be made. To be specific, we assume the process by which this particle P_0 affects the other particles in stages, without at first considering the time at which the

collisions take place. The first stage in this process is the first collision of particle P_0 with one other particle P_1, after which this pair of particles are both affected by the initial coordinate and momentum (state) of the particle P_0. The subsystem S_1 after this first stage consists of both particles of the pair.

In the second stage of the process, each particle of S_1 collides with another particle, assumed to be different, giving the four particles of subsystem S_2 affected by the initial state of P_0. Notice that the two collisions of the second stage need not occur at the same time: questions of timing are considered later.

Every particle of subsystem S_n belongs to all later subsystems. We also assume for simplicity that every particle of S_n collides with a particle which is *not* in S_n, so that the number of particles involved doubles at each stage, and the number in subsystem S_n is 2^n.

For every collision one of the colliding particles belongs to the previous subsystem S_{n-1}, before the collision, and also to the subsystem S_n after the collision. One of the particles is new, and belongs only to S_n. Starting with particle P_0, we can follow a sequence of collisions and particles leading to a particle $P_j^{(n)}$ of S_n. For some of these collisions the particle in this sequence after the collision will be the same as the particle before the collision. These collisions are said to be *direct*. For others, the particle leaving the collision will be one of the new ones, and so it will be a different particle than the one that entered. These collisions are said to be *switch* collisions. In a typical sequence, the number of direct and switch collisions is roughly equal.

Now consider the gas dynamics.

3.2 Dynamics of the Arnold gas

First consider the dynamics of the first collision. Let $\mathbf{X}_0$ and $\mathbf{X}_1$ be the initial state (coordinate and momentum) of P_0 and P_1, and let $\mathbf{X}'_0$ and $\mathbf{X}'_1$ be the final states of these particles.

Let M be the matrix of the Arnold cat map [14]:

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \tag{1}$$

with eigenvalues

$$\lambda_\pm = \frac{3 \pm \sqrt{5}}{2}. \tag{2}$$

Then the equations of the collision, in terms of the centers of mass and the relative coordinates are

$$\mathbf{X}'_0 + \mathbf{X}'_1 = \mathbf{X}_0 + \mathbf{X}_1, \qquad \mathbf{X}'_0 - \mathbf{X}'_1 = M(\mathbf{X}_0 - \mathbf{X}_1) \tag{3}$$

and in terms of the states of the individual particles are

$$\mathbf{X}'_0 = \frac{(I+M)\mathbf{X}_0}{2} + \frac{(I-M)\mathbf{X}_1}{2} = K_+\mathbf{X}_0 + K_-\mathbf{X}_1, \tag{4}$$

$$\mathbf{X}'_1 = \frac{(I-M)\mathbf{X}_0}{2} + \frac{(I+M)\mathbf{X}_1}{2} = K_-\mathbf{X}_0 + K_+\mathbf{X}_1 \tag{5}$$

In this collision the linear dependence of the final state of a particle on its initial state is given by the matrix $K_+ = (I+M)/2$. This is the *direct* matrix. The dependence of the final state of a particle on the initial state of the other particle is given by the *switch* matrix $K_- = (I-M)/2$.

Using direct and switch matrices we can obtain the linear dependence of the state of any particle on the initial state X_0 of P_0. For a particle $P_j^{(n)}$ of subsystem S_n, it has the form

$$\mathbf{X}_j^{(n)} = (K^+)^{n_1}(K^-)^{n_2}\mathbf{X}_0 + \mathbf{Y}_0, \tag{6}$$

where

$$n_1 + n_2 = n, \tag{7}$$

Here n_1 is the number of direct matrices and n_2 is the number of switch matrices in the sequence of particles starting with P_0 and finishing with $P_j^{(n)}$. $\mathbf{Y}_0$ is independent of $\mathbf{X}_0$ and represents the initial states of all the other particles of S_n.

The value of n_1 and thus of n_2 depends on the particle $P_j^{(n)}$. If it is the same particle as P_0, then there are no switches and $n_1 = n$, $n_2 = 0$. If the sequence of particles is a new particle at every stage, from the beginning to the end, then there are n switches and $n_1 = 0$, $n_2 = n$. The others lie between these two extremes. The number of times a pair (n_1, n_2) occurs is given by the number of switches, and this forms a binomial distribution, so the mean values are given by

$$n_1/n \approx 1/2 \approx n_2/n \qquad \text{(mean values)}, \tag{8}$$

and the deviation from this mean becomes small as n increases.

3.3 Bounds on dilation factors

Because every collision is represented by a linear map, the same linear relations hold for displacements $\Delta\mathbf{X}$ in $\mathbf{X}$ as for $\mathbf{X}$ itself, except for additive constants like $\mathbf{Y}_0$. So if the initial state of P_0 is displaced by $\Delta\mathbf{X}_0$, then the corresponding displacement in $\Delta\mathbf{X}_j^{(n)}$ is given by

$$\Delta\mathbf{X}_j^{(n)} = (K^+)^{n_1}(K^-)^{n_2}\Delta\mathbf{X}_0, \tag{9}$$

where it is assumed that the initial state of every other particle is held constant.

For a single operation of the Arnold cat map M, the eigenvalues $\lambda_\pm$ and corresponding normalized eigenvectors $\xi_\pm$ are given by

$$M\xi_\pm = \lambda_\pm \xi_\pm, \qquad \lambda_\pm = \tfrac{1}{2}(3 \pm \sqrt{5}). \tag{10}$$

The dilation of the displacement is given by

$$\left| \frac{\Delta \mathbf{X}_j^{(n)}}{\Delta \mathbf{X}_0} \right|, \tag{11}$$

and this depends on the direction of $\Delta \mathbf{X}_0$. For simplicity, suppose it is in the direction of ξ_+, so that

$$\Delta \mathbf{X}_0 = \epsilon \xi_+, \tag{12}$$

where ϵ is the amplitude of the initial displacement. Now ξ_+ is an eigenvector of K^+ and of K^- as defined in equation (3),

$$K^+ \xi_+ = k^+ \xi_+ \tag{13}$$
$$K^- \xi_+ = k^- \xi_+. \tag{14}$$

and the corresponding eigenvalues are given by

$$k^+ = \frac{1+\lambda_+}{2} = \frac{5+\sqrt{5}}{4} \tag{15}$$
$$k^- = \frac{1-\lambda_+}{2} = -\frac{1+\sqrt{5}}{4} \tag{16}$$

We also need

$$|k^+ k^-| = 1 + \frac{3}{8}(\sqrt{5}-1) \approx 1.46. \tag{17}$$

The approximate mean dilation for the displacement of a single particle after n collisions is therefore

$$\left| \frac{\Delta \mathbf{X}_j^{(n)}}{\Delta \mathbf{X}_0} \right| \approx |k^+ k^-|^{n/2} \approx 1.46^{n/2} \approx 1.2^n > 1 \tag{18}$$

The important thing to notice here is that the mean displacement for the state of any particle of a subsystem at any later time is greater than the original displacement for the state of P_0.

Now consider these changes as changes in the state of the whole gas. The initial displacement has magnitude ϵ. The final displacement in the phase space of the entire gas has a magnitude equal to the square root of the sum of the squares of the displacements of each particle. The number of particles of S_n is 2^n, so the dilation for the whole gas is bounded below by the inequality

$$\text{dilation for gas} > \sqrt{2^n} = 2^{n-1}. \tag{19}$$

The magnitude of the displacement in the phase space of the whole gas of N particles more than doubles at each stage, and becomes significant after fewer than $1 + \log_2 N$ collisions, in the sense described in the next section.

Now we come to the question of times. The stages correspond to different times for different collisions, but the time for n collisions is roughly the same when n is sufficiently large, and approximately equal to $n\Delta t$ where Δt is the mean time between two collisions of a single particle.

The time T_s for the displacement to become significant, in the sense that a typical particle of the gas of N particles at time $t = T$ has roughly the same displacement as P_0 has at time $t = 0$ is then

$$T_s \approx \Delta t \log_2 N \tag{20}$$

4 Fluctuation of the density

This section is devoted to discuss the fluctuation of the Arnold gas density in phase space. An exponent for the Fourier components of the density is defined. This exponent gives a means of characterizing the fluctuation of the density.

The phase density for a system comprised of N particles in a unit square is given by

$$n(\mathbf{X}, t) = \sum_{i=1}^{N} \delta(\mathbf{X}(t) - \mathbf{X}_i(t)) \tag{21}$$

where $\mathbf{X}_i(t) = [x_i(t), p_i(t)]$ and $x_i(t)$ and $p_i(t)$ are the position and momentum of ith particle, respectively. It is easy to see that

$$\int n(\mathbf{X}, t) d\mathbf{X} = \sum_{i=1}^{N} \int \delta(\mathbf{X} - \mathbf{X}_i) d\mathbf{X} = N \tag{22}$$

The Fourier expansion of $n(\mathbf{X}, t)$ is given by:

$$n(\mathbf{X}, t) = \sum_{i=1}^{N} \delta(\mathbf{X}(t) - \mathbf{X}_i(t)) = L^{-2} \sum_{\mathbf{k}} n_{\mathbf{k}}(t) \exp(i\mathbf{k} \cdot \mathbf{X}(t)) \tag{23}$$

with

$$n_{\mathbf{k}}(t) = \int n(\mathbf{X}, t) \exp\left[-i\mathbf{k}\cdot\mathbf{X}\right] d\mathbf{X} = \sum_{i=1}^{N} \exp[-i\mathbf{k}\cdot\mathbf{X}_i(t)] \tag{24}$$

where L (We choose $L = 1$) is the length of the square containing N particles and $\sum_{\mathbf{k}}$ stands for the sum over all possible discrete values of $\mathbf{k}$ allowed by the imposed boundary condition.

Now we consider the probability density $\tilde{n}(\mathbf{X}, t)$:

$$\tilde{n}(\mathbf{X}, t) = \frac{1}{N} n(\mathbf{X}, t) \tag{25}$$

We are now in the position to compute a bound on the ratio of the Fourier component $\tilde{n}_{\mathbf{k}}(t)$ and the initial displacement of a particle, say, particle P_0. To do so, first, note that

$$\tilde{n}_{\mathbf{k}}(t) = \frac{1}{N} n_{\mathbf{k}}(t) \tag{26}$$

Hence, $\Delta\tilde{n}_{\mathbf{k}}(t) = \frac{1}{N}\Delta n_{\mathbf{k}}(t)$. Note that t is an integer representing the number of iterations. So the ratio is given by

$$\left|\frac{\Delta\tilde{n}_{\mathbf{k}}(t)}{\Delta\mathbf{X}_0}\right| = \left|\frac{\sum_i e^{i\mathbf{k}\cdot\mathbf{X}_i^{(t)}}\mathbf{k}\cdot\Delta\mathbf{X}_i^{(t)}}{N\Delta\mathbf{X}_0}\right| \tag{27}$$

Note that

$$\mathbf{k}\cdot\Delta\mathbf{X}_i^{(t)} = \mathbf{k}\cdot(K^+)^{n_1}(K^-)^{n_2}\Delta\mathbf{X}_0. \tag{28}$$

If we consider the displacement of the particle P_0 in the direction of ξ_+: $\Delta\mathbf{X}_0 = \epsilon\xi_+$, then the equation (28) becomes

$$\mathbf{k}\cdot\Delta\mathbf{X}_j^{(t)} \approx (1.2)^t\epsilon\mathbf{k}\cdot\xi_+ \tag{29}$$

An exponent for the Fourier component $\tilde{n}_{\mathbf{k}}(t)$ can be defined as

$$\lambda = \ln\left|\frac{\sum_i e^{i\mathbf{k}\cdot\mathbf{X}_i^{(t)}}(\mathbf{k}\cdot\xi_+)}{N}\right|^{\frac{1}{t}} + \ln(1.2). \tag{30}$$

Note that the exponent λ plays the similar role to the Lyapunov exponent. But unlike the Lyapunov exponent, λ is not always positive. For sufficiently large t, the second term in Eq. (30) is dominant. So we have

$$\lambda \approx \ln(1.2) \approx 0.18 > 0 \tag{31}$$

Hence, in the long time limit, with (30) and (31), the dilation of the Fourier component $\tilde{n}_{\mathbf{k}}(t)$ can be written as

$$\Delta\tilde{n}_{\mathbf{k}}(t) \approx \epsilon e^{\lambda t}. \tag{32}$$

It should be emphasized that the exponent λ may not be positive at the early stage as can be seen from (30). It only becomes positive when the collisions have significantly influenced the whole gas. This is consistent with our expectation.

It is interesting to make some rough estimations of the time scale. We assume that gas at room temperature($T \approx 300K$) and atmosphere pressure ($p \approx 10^6$dynes/cm^2) is contained in a square with area 1cm^2. The number of Arnold gas molecules is about 2.5×10^{19}. The mean free path is $l_m = 2 \times 10^{-5}$cm. The mean speed of molecules is $v_m = 4 \times 10^4$cm/sec. Then the mean free time is $t_m = l_m/v_m \approx 5 \times 10^{-10}$ sec . Now we see that in one second, there are approximately 2×10^9(iterations) collisions. We see that the small changes in the phase space of a single particle can produce the exponential difference in the particle probability density $\Delta\tilde{n}_{\mathbf{k}}(t) \approx \epsilon e^{\lambda t}$.

Finally, let's take a look at the familiar example of light scattering. For a volume element ΔV which is of the dimension of the order of the wavelength of visible light($\approx 5^{-5}$cm), the fluctuation are significant, as shown above. The Rayleigh scattering is due to the fluctuation of the particle density. This is in turn responsible for the blue of the sky.

5 Concluding comments

We have constructed a classical model of an equilibrium gas to represent the classical stage of a general quantum measurement. An exponent is used to characterize the fluctuation of the gas density relative to the initial displacement of a single particle. To be specific, we have shown that density of the Arnold gas is highly sensitive to a disturbance of the initial position and momentum of one particle.

In some sense, the model looks very artificial, because of the following differences between the model gas and a real gas of molecules:

(i) Normal gases are 3-dimensional, not 1-dimensional, and the centre of mass of a molecule of a gas has a 6-dimensional, not a 2-dimensional, phase space.

(ii) An ordinary collision between two molecules of a gas does not resemble any kind of linear cat map, even a 6-dimensional cat map. A collision of the cat map here corresponds to a collision and subsequent drift in a real gas.

(iii) After a sufficient number of collisions, the number of molecules in the system S_n affected by a displacement of P_0 does not double at each stage, because molecules of a subsystem can collide with each other. The number of molecules in S_{n+1} is then less than 2^{n+1}. Some of the particles of the subsystem are affected as a result of two or more sequences of collisions, between different particles. For a real gas like the atmosphere, ignoring the effects of radiation, the particles in the subsystem affected by a displacement

$\Delta \mathbf{X}_0$ is determined by the speed of sound, and increases asymptotically as the cube of the time. Because the number of particles in the real gas is less than for the model gas, the dilation factor is larger for the real gas.

If the displacement in the phase space of the two particles after a collision is more than (not necessarily more than double) the displacement in the phase space of one of the particles before the collision, when the displacement of the other particle is zero, then the displacement of the phase point of the gas grows exponentially at each stage. A typical ratio is more difficult to work out for the nonlinear dynamics of real collisions, partly because 'collision' is not clearly defined for potentials of infinite range.

For a particle which receives a displacement as a result of two different sequences of collisions, it may be a good approximation to assume that these displacements are statistically independent, in which case the resultant displacement is equivalent to displacements of different molecules.

The details of these considerations go beyond the scope of this paper. The present paper is a first step towards the general quantum measurement theory of equilibrium systems. Of course, the present paper is not complete because we have ignored the relation between quantum fluctuations and classical fluctuations at the ambiguous boundary between the 'classical' and the 'quantum' domains.

There are many situations in which a general quantum measurement is of interest. A very remarkable example of this situation arises in the early universe context in which the density fluctuation is important for the early evolution of the universe. Crudely speaking, the long wavelength radiation could serve as the environment field whereas the short wavelength as quantum modes [15, 16, 17]. The interaction between those different modes will be important for the development of early universe such as vacuum particle creation and structural formation.

One of John Bell's major objections to quantum 'measurement' is overcome by generalizing the definition to include processes in the big world. With this definition, quantum measurement takes place in those equilibrium systems for which the classical motion is chaotic, even though the measurement cannot be used in that case to get detailed information about individual quantum states. Consequently the dynamics of quantum measurement has universal significance and so have its properties.

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